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Interpolation by Multidimensional Periodic Splines

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This paper is concerned with interpolation by multidimensional periodic splines associated with certain elliptic differential operators. We develop an a priori error estimate for the solution obtained by spline interpolation. Finally we investigate the problem of uniform approximation by multidimensional periodic splines (as basis funnctions).

INTRODUCTION

Let $g^1, ..., g^q$ be a basis of a non-degenerate lattice Λ in Euclidean space $\mathbb{R}^q$. Denote by $\mathscr{F}$ the half-open parallelotope consisting of all points $x \in \mathbb{R}^q$ of the form

$$x = \sum_{i=1}^{q} t_i g^i \tag{0.1}$$

$(-1/2 \leqslant t_i < 1/2,\ i = 1, ..., q)$. $\mathscr{F}$ is called the fundamental cell of the lattice Λ. The volume of this cell is

$$\| \mathscr{F} \| = |\det(g^1, ..., g^q)|. \tag{0.2}$$

If Λ is a non-degenerate lattice in $\mathbb{R}^q$, then the set of all $h \in \mathbb{R}^q$ such that gh is an integer for all $g \in \Lambda$ is again a lattice called the inverse lattice Λ^{-1}. Obviously, $(\Lambda^{-1})^{-1} = \Lambda$ (cf. [1, 12]).

The functions ϕ_h, $h \in \Lambda^{-1}$, defined by

$$\phi_h(x) = \frac{1}{\sqrt{\| \mathscr{F} \|}} e(hx), \qquad e(hx) = e^{2\pi i(hx)}, \tag{0.3}$$

are Λ-periodic, i.e., $\phi_h(x+g) = \phi_h(x)$ for all $g \in \Lambda$. An elementary calculation yields

$$(-\Delta_x - \lambda_h)\, \phi_h(x) = 0, \qquad \lambda_h = 4\pi^2 h^2 \tag{0.4}$$

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(Δ: Laplace-operator). The functions ϕ_h, $h \in \Lambda^{-1}$, are the only twice continuously differentiable eigenfunctions corresponding to the eigenvalues λ_h of the Laplace-operator for the "boundary condition" of Λ-periodicity. The system $\{\phi_h \mid h \in \Lambda^{-1}\}$ is orthonormal in the sense of the L^2-inner product

$$(\phi_h, \phi_{h'})_{L^2(\mathscr{F})} = \int_{\mathscr{F}} \phi_h(x)\, \overline{\phi_{h'}(x)}\, \mathrm{dV} = \delta_{hh'} = \begin{cases} 1 & h = h' \\ 0 & h \neq h' \end{cases} \tag{0.5}$$

(dV: volume element). The correspondence $f \leftrightarrow (f, \phi_h)_{L^2(\mathscr{F})}$ is a unitary mapping of $L^2(\mathscr{F})$ onto $l^2(\Lambda^{-1})$ (cf. [17]).

Let ρ be a non-negative real. Then there only exist a finite number $m = m_\rho$ of non-negative reals $r \leqslant \rho$ with

$$\sum_{\substack{|h| = r \\ h \in \Lambda^{-1}}} 1 > 0. \tag{0.6}$$

Denote all these non-negative reals $\leqslant \rho$ satisfying (0.6) by $r_1, \ldots r_m$. We let

$$M = M_\rho = \sum_{\substack{|h| \leqslant \rho \\ h \in \Lambda^{-1}}} 1 = \sum_{j=1}^{m} \sum_{\substack{|h| = r_j \\ h \in \Lambda^{-1}}} 1. \tag{0.7}$$

M is the total number of lattice points $h \in \Lambda^{-1}$ on and inside the sphere around the origin with radius ρ. $\Lambda_N^{-1} = \Lambda_{\rho,N}^{-1}$ denotes a set of $N \geqslant M$ elements $h \in \Lambda^{-1}$ containing all $h \in \Lambda^{-1}$ with $|h| \leqslant \rho$ as subset. Clearly, we have

$$\Lambda_M^{-1} = \{h \in \Lambda^{-1} \mid |h| \leqslant \rho\}. \tag{0.8}$$

M is the dimension of the space $\mathscr{P} = \mathscr{P}_\rho$ of all linear combinations of the functions ϕ_h, $h \in \Lambda_M^{-1}$, in $\mathbb{R}^q$

$$\mathscr{P} = \operatorname{span}(\phi_h)_{h \in \Lambda_M^{-1}}. \tag{0.9}$$

Consider a $\mathscr{P}$-unisolvent set $X_M = \{x_h \in \mathscr{F} \mid h \in \Lambda_M^{-1}\}$, i.e., a set of points $x_h \in \mathscr{F}$ such that the rank of the (M, M)-matrix

$$(\phi_h(x_{h'}))_{h, h' \in \Lambda_M^{-1}} \tag{0.10}$$

is equal to M (cf. [10]). Then we are able to interpolate a given set $\{\gamma_h \in \mathbb{C} \mid h \in \Lambda_M^{-1}\}$ by a unique $P \in \mathscr{P}$: $P(x_h) = \gamma_h$, $h \in \Lambda_M^{-1}$. However, for any set $X_N = \{x_h \in \mathscr{F} \mid h \in \Lambda_N^{-1}\}$ containing X_M as a proper subset (so that $N > M$), this interpolation property cannot be guaranteed in general. In this case we are led in a canonical way to a spline interpolation problem, namely the following. We consider the differential operator

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m} \tag{0.11}$$

with

$$\partial_j^{\alpha_j} = (-\Delta - 4\pi^2 r_j^2)^{\alpha_j}, \tag{0.12}$$

where $\alpha = (\alpha_1, ..., \alpha_m)$ is a multiindex, i.e., an m-tuple of positive integers $\alpha_1, ..., \alpha_m$. Hence, $[\alpha] := \alpha_1 + \cdots + \alpha_m \geq m$. The kernel of ∂^α is the linear space $\mathscr{P}$.

Within the (Sobolev-like) space $\mathscr{H}$ of Λ-periodic functions U with square-integrable derivatives $\partial^{\alpha/2} U = (\partial^\alpha)^{1/2}\, U$ we look for the solution S_N of the minimization problem

$$\int_{\mathscr{F}} |\partial^{\alpha/2} S_N(x)|^2 \,\mathrm{dV} = \inf_{U \in \mathscr{I}_N} \int_{\mathscr{F}} |\partial^{\alpha/2}\, U(x)|^2 \,\mathrm{dV} \tag{0.13}$$

in the set $\mathscr{I}_N$ of all $\mathscr{H}$-interpolants to the given data

$$\mathscr{I}_N = \{ U \in \mathscr{H} \mid U(x_h) = \gamma_h, h \in \Lambda_N^{-1} \}. \tag{0.14}$$

This procedure is reasonable because it gives an interpolant which is as close to being a polynomial $P \in \mathscr{P}$ as can be achieved for the N data. In addition, the spline interpolant is the "smoothest" in the sense of the problem (0.13), (0.14), thereby avoiding wild oscillations in the interpolant. Moreover, the previous case when $N = M$ also is included since the uniquely defined polynomial in $\mathscr{P}$ is obviously the solution of the minimation problem (0.13), (0.14).

In this paper we are concerned with the problem of interpolating multidimensional periodic functions by periodic splines and developing an a priori error bound for the approximation. The contents of the paper are organized as follows: in the first section the structure of the (Sobolev-like) space $\mathscr{H}$ is discussed in more detail. In the second section we deal with the interpolation problem using multidimensional periodic splines. By analogy with the approach known from surface spline theory (cf. [13]) the interpolation process can be made surprisingly simple and reasonably efficient for numerical purposes. The third section states and proves an a priori estimate in problems of uniform approximation of "$\mathscr{H}$-smooth" functions by periodic splines interpolating at prescribed knots. Our paper ends with some remarks about uniform approximation of continuous, Λ-periodic functions by splines. It is shown that any continuous, Λ-periodic function in $\mathbb{R}^q$ can be approximated uniformly to any given accuracy by Λ-periodic splines in such a way that the approximation error vanishes at a prescribed finite set of knots (see, in comparison, e.g. [11]).

The periodic splines discussed here turn out to be multidimensional generalizations of the periodic splines on the circle (cf. [15] for the classical approach, [16] and the references therein for recent developments) and natural analogues to the splines on the sphere (cf. [4, 5, 6, 8, 19]). Our spline concept is based on the (iterated) Laplace-operator and on arbitrary (non-degenerate) lattices. Therefore all spline representations are included which are based on (iterations of) arbitrary second order elliptic differential operators with constant coefficients.

1. The Space $\mathscr{H}$

Let $\mathscr{H}$ be the linear space of all Λ-periodic distributions (i.e., Λ-periodic, continuous linear functionals on the space $\mathscr{E}$ of Λ-periodic infinitely differentiable functions in $\mathbb{R}^q$, provided with the canonical topology) for which $\partial^{\alpha/2}U$ (in the distributional sense) is square-integrable on $\mathscr{F}$

$$\mathscr{H} = \{U \in \mathscr{E}' \mid \partial^{\alpha/2} U \in L^2(\mathscr{F})\}, \tag{1.1}$$

it always being understood that $[\alpha] > q/2$. $\mathscr{H}$ is naturally equipped with the semi-inner product $(\cdot, \cdot)_{\mathscr{H}}$ corresponding to the semi-norm

$$|U|_{\mathscr{H}} = \left\{ \int_{\mathscr{F}} |\partial^{\alpha/2} U(x)|^2 \, dV \right\}^{1/2}, \tag{1.2}$$

where $\partial^{\alpha/2}$ is to be interpreted in the distributional sense. The kernel of this (Sobolev-like) semi-norm $|\cdot|_{\mathscr{H}}$ is the linear space $\mathscr{P}$.

By proceeding essentially as explained in [13] the following can be proved (see also [2]).

Theorem 1. *The semi-normed space $\mathscr{H}$, defined by* (1.1) *and* (1.2), *is a functional semi-Hilbert subspace of the space C_Λ of Λ-periodic, continuous functions in $\mathbb{R}^q$.*

Consider a $\mathscr{P}$-unisolvent set $X_M = \{x_h \in \mathscr{F} \mid h \in \Lambda_M^{-1}\}$. Then there exists in $\mathscr{P}$ a unique basis $\{B_h \mid h \in \Lambda_M^{-1}\}$ given by

$$B_h(x) = \sum_{h' \in \Lambda_M^{-1}} C_{h'}^h \phi_{h'}(x), \qquad x \in \mathbb{R}^q, \tag{1.3}$$

such that

$$B_h(x_{h'}) = \delta_{hh'}, \qquad h, h' \in \Lambda_M^{-1}. \tag{1.4}$$

For every $U \in \mathscr{H}$, the unique $\mathscr{P}$-interpolant pU of U on the $\mathscr{P}$-unisolvent set X_M under consideration is given by the "Lagrange formula"

$$pU = \sum_{h \in \Lambda_M^{-1}} U(x_h)\, B_h. \tag{1.5}$$

The mapping $p: \mathscr{H} \to \mathscr{H}$ is a continuous linear projector of $\mathscr{H} \subset C_\Lambda$ onto $\mathscr{P}$. Hence, p determines the following direct sum decomposition

$$\mathscr{H} = \mathscr{P} \oplus \mathring{\mathscr{H}}, \tag{1.6}$$

where

$$\mathring{\mathscr{H}} = \{ U \in \mathscr{H} \mid U(x_h) = 0, h \in \Lambda_M^{-1} \}. \tag{1.7}$$

That means that any $U \in \mathscr{H}$ can be represented uniquely in the form $U = pU + \mathring{U}$, $\mathring{U} \in \mathring{\mathscr{H}}$. The space $\mathring{\mathscr{H}}$, as defined by (1.7) equipped with the inner product corresponding to (1.2), is a functional Hilbert space (it is indeed isometrically isomorphic to $\mathscr{H}$).

Since $\mathring{\mathscr{H}}$ is a functional Hilbert space of Λ-periodic, continuous functions, for each (fixed) $y \in \mathscr{F}$, the evaluation function $\delta_y: \mathring{U} \to \mathring{U}(y)$ on $\mathring{\mathscr{H}}$ is bounded. That is, δ_y can be regarded as the Dirac measure at the point $y \in \mathscr{F}$, and the following representation formula in $\mathring{\mathscr{H}}$ holds true

$$\mathring{U}(y) = \langle \mathring{U}, \delta_y \rangle = (\mathring{U}, \mathring{K}_y^\alpha)_{\mathscr{H}}, \qquad \mathring{U} \in \mathring{\mathscr{H}}, \tag{1.8}$$

where $\mathring{K}_y^\alpha \in \mathring{\mathscr{H}}$ is the Riesz-representer of δ_y and $\langle \cdot, \cdot \rangle$ denotes as usual the duality bracket between dual topological vector spaces. Consequently, for all $\psi \in \mathscr{E}$, we have

$$\psi(y) - p\psi(y) = \langle \mathring{\psi}, \partial^\alpha \mathring{K}_y^\alpha \rangle = (\psi - p\psi, \mathring{K}_y^\alpha)_{\mathscr{H}}. \tag{1.9}$$

On the other hand it follows for all $\psi \in \mathscr{E}$ that

$$\psi(y) - p\psi(y) = \langle \mathring{\psi}, {}^M\delta_y - \sum_{h \in \Lambda_M^{-1}} \overline{B_h(y)}\, {}^M\delta_{x_h} \rangle, \tag{1.10}$$

where

$${}^M\delta_y = \delta_y - \sum_{h \in \Lambda_M^{-1}} \overline{\phi_h(y)}\, \phi_h. \tag{1.11}$$

By comparison of (1.9) and (1.10) we see that K_y^α satisfies the distributional equation

$$\partial^\alpha \mathring{K}_y^\alpha = {}^M\delta_y - \sum_{h \in \Lambda_M^{-1}} \overline{B_h(y)}\, {}^M\delta_{x_h} \tag{1.12}$$

in $\mathring{\mathscr{H}} \subset \mathscr{E}'$. The distributional equation

$$\partial^\alpha G_y^\alpha = {}^M\delta_y \tag{1.13}$$

is solvable in $\mathscr{H}$ uniquely apart from an additive element of $\mathscr{P}$ by Green's (lattice) function of the operator ∂^α (cf. [7, 9])

$$G_y^\alpha(x) = G^\alpha(x, y) = \sum_{h \notin \Lambda_M^{-1}} |\lambda_h^\alpha|^{-1} \overline{\phi_h(y)}\, \phi_h(x), \tag{1.14}$$

where λ_h^α, $h \in \Lambda_M^{-1}$, is given by

$$\lambda_h^\alpha = [(4\pi^2 h^2 - 4\pi^2 r_1^2)^{\alpha_1} \cdots (4\pi^2 h^2 - 4\pi^2 r_m^2)^{\alpha_m}] \tag{1.15}$$

and the series on the right hand side of (1.14) is extended over all $h \in \Lambda^{-1}$ for which $h \notin \Lambda_M^{-1}$. In fact, by techniques known in potential theory (cf., e.g., [9]) it can be shown that

$$\partial^\beta G^\alpha(x, y) = \begin{cases} O(|x-y|^{2([\alpha]-[\beta])-q} \ln |x-y|) & \text{if } 2([\alpha]-[\beta]) \geqslant q,\ q \text{ even} \\ O(|x-y|^{2([\alpha]-[\beta])-q} & \text{otherwise} \end{cases} \tag{1.16}$$

provided that $\beta = (\beta_1, ..., \beta_m)$ is a multiindex of non-negative integers $\beta_1, ..., \beta_m$ with $\beta_j \leqslant \alpha_j$, $j = 1, ..., m$ and $[\beta] < [\alpha]$.

Apart from an additive element of $\mathscr{P}$, the distribution $H_y^\alpha \in \mathscr{H}$ given by

$$H_y^\alpha(x) = H^\alpha(x, y) = G^\alpha(x, y) - \sum_{h \in \Lambda_M^{-1}} \overline{B_h(y)}\, G^\alpha(x, x_h) \tag{1.17}$$

is the unique solution of the distributional equation

$$\partial^\alpha H_y^\alpha = {}^M\delta_y - \sum_{h \in \Lambda_M^{-1}} \overline{B_h(y)}\, {}^M\delta_{x_h}. \tag{1.18}$$

But this means that $\mathring{K}_y^\alpha \in \mathring{\mathscr{H}}$ given by

$$\begin{aligned} \mathring{K}_y^\alpha(x) &= \mathring{K}^\alpha(x, y) \\ &= G^\alpha(x, y) - \sum_{h \in \Lambda_M^{-1}} \overline{B_h(y)}\, G^\alpha(x, x_h) \\ &\quad - \sum_{h \in \Lambda_M^{-1}} G^\alpha(x_h, y)\, B_h(x) + \sum_{h \in \Lambda_M^{-1}} \sum_{h' \in \Lambda_M^{-1}} \overline{B_h(y)}\, G^\alpha(x_{h'}, x_h)\, B_{h'}(x) \end{aligned} \tag{1.19}$$

is the unique element in $\mathring{\mathscr{H}}$ satisfying (1.12).

Summarizing our results we therefore obtain

THEOREM 2. *The space* $\mathring{\mathscr{H}}$, *defined by* (1.7) *with norm* (1.2), *is a functional Hilbert subspace of* C_Λ. $\mathring{K}^\alpha(\cdot,\cdot)$, as *defined by* (1.19), *is the reproducing kernel for* $\mathring{\mathscr{H}}$, *i.e.*, (i) *for each fixed* $y \in \mathbb{R}^q$, $\mathring{K}^\alpha(\cdot, y)$ *is in* $\mathring{\mathscr{H}}$, (ii) *for every function* $\mathring{U} \in \mathring{\mathscr{H}}$ *and for every* $y \in \mathbb{R}^q$, *the reproducing property*

$$\mathring{U}(y) = \int_{\mathscr{F}} \partial^{\alpha/2}\, \mathring{U}(x)\, \overline{\partial^{\alpha/2} \mathring{K}^\alpha(x, y)}\, \mathrm{dV}(x)$$

holds.

2. Spline Interpolation

A set $X_N = \{x_h \in \mathscr{F} \mid h \in \Lambda_N^{-1}\}$ of $N \geqslant M$ points $x_h \in \mathscr{F}$ is called a $\mathscr{P}$-*admissible set* if it contains a $\mathscr{P}$-unisolvent system $X_M = \{x_h \in \mathscr{F} \mid h \in \Lambda_M^{-1}\}$ as subset.

DEFINITION 1. Given a $\mathscr{P}$-admissible system $X_N = \{x_h \in \mathscr{F} \mid h \in \Lambda_N^{-1}\}$, then any function $S \in \mathring{\mathscr{H}}$ of the form

$$S(x) = \sum_{h \in \Lambda_M^{-1}} a_h B_h(x) + \sum_{h \in \Lambda_N^{-1} - \Lambda_M^{-1}} a_h \mathring{K}^\alpha(x, x_h), \qquad a_h \in \mathbb{C},\ x \in \mathbb{R}^q, \tag{2.1}$$

is called a Λ-periodic $\mathscr{P}$-spline in $\mathscr{H}$ relative to X_N. (For $N = M$, S reduces to the first sum of the right hand side of (2.1).)

The space $\mathscr{S} = \mathscr{S}_{\mathscr{P}}^{\Lambda}(X_N)$ of all Λ-periodic $\mathscr{P}$-splines in $\mathscr{H}$ relative to X_N is an N-dimensional linear subspace of $\mathscr{H}$ containing the class $\mathscr{P}$.

THEOREM 3. *Any* $S \in \mathscr{S}$ *can be represented in the form*

$$S(x) = P(x) + \sum_{h \in \Lambda_N^{-1}} b_h G^\alpha(x, x_h), \qquad x \in \mathbb{R}^q,\ P \in \mathscr{P}, \tag{2.2}$$

where the coefficients $b_h \in \mathbb{C}$ *have to satisfy the linear equations*

$$0 = \sum_{h \in \Lambda_N^{-1}} b_h \phi_{h'}(x_h), \qquad h' \in \Lambda_M^{-1}. \tag{2.3}$$

The proof follows easily from the definition of Λ-periodic $\mathscr{P}$-splines by straightforward calculation. Therefore it is obvious that the Λ-periodic splines discussed here form multidimensional generalizations of the one-dimensional trigonometric splines due to Schoenberg [15].

Suppose now that there are given N prescribed data points (x_h, γ_h), $h \in \Lambda_N^{-1}$, corresponding to a $\mathscr{P}$-admissible system $X_N = \{x_h \in \mathscr{F} \mid h \in \Lambda_N^{-1}\}$.

We consider the problem of finding the smoothest function in the set

$$\mathscr{I}_N = \{U \in \mathscr{H} \mid U(x_h) = \gamma_h,\ h \in \Lambda_N^{-1}\}$$

of all $\mathscr{H}$-interpolants to the data, where by "smoothest" we mean that the semi-norm $|\cdot|_{\mathscr{H}}$ is minimized in $\mathscr{H}$.

For that purpose we need some preliminaries formulated in the following lemmata.

LEMMA 1. *If* $U \in \mathscr{I}_N$ and $S \in \mathscr{S}$, *then*

$$(U, S)_{\mathscr{H}} = \sum_{h \in \Lambda_N^{-1} - \Lambda_M^{-1}} a_h \left[\gamma_h - \sum_{h' \in \Lambda_M^{-1}} \gamma_{h'} B_{h'}(x_h)\right].$$

LEMMA 2. *There exists a unique* $S \in \mathscr{S} \cap \mathscr{I}_N$. *Denote this spline briefly by* S_N.

Proof. Any spline $S \in \mathscr{S}$ of the form (2.1) contains a total of N coefficients $a_h \in \mathbb{C}$, $h \in \Lambda_N^{-1}$. Thus, $S(x_{h'}) = \gamma_{h'}$, $h' \in \Lambda_N^{-1}$, is equivalent to the linear equations

$$\sum_{h \in \Lambda_N^{-1} - \Lambda_M^{-1}} a_h \mathring{K}^\alpha(x_{h'}, x_h) = \gamma_{h'} - \sum_{h \in \Lambda_M^{-1}} \gamma_h B_h(x_{h'}), \qquad h' \in \Lambda_N^{-1} - \Lambda_M^{-1}. \tag{2.4}$$

The coefficient matrix is (Hermitian) symmetric and positive definite as Gram matrix of a sequence of linearly independent elements in $\mathscr{H}$. Hence, the linear system (2.4) is uniquely solvable. ∎

The solution can be obtained by standard algorithms based on the idea of Cholesky's factorization (cf., e.g., [3, 13]).

LEMMA 3. *If* $U \in \mathscr{I}_N$, *then*

$$\int_{\mathscr{F}} |\partial^{\alpha/2} U(x)|^2 \, dV = \int_{\mathscr{F}} |\partial^{\alpha/2} S_N(x)|^2 \, dV + \int_{\mathscr{F}} |\partial^{\alpha/2}(S_N(x) - U(x)|^2 \, dV.$$

THEOREM 4. *The interpolation problem*

$$\int_{\mathscr{F}} |\partial^{\alpha/2} S_N(x)| \, dV = \inf_{U \in \mathscr{I}_N} \int_{\mathscr{F}} |\partial^{\alpha/2} U(x)|^2 \, dV$$

is well posed in the sense that its solution exists, is unique, and depends continuously on the data γ_h, $h \in \Lambda_N^{-1}$.

3. Error Estimate

For any $\mathscr{P}$-admissible set $X_N = \{x_h \in \mathscr{F} \mid h \in \Lambda_N^{-1}\}$ there exists the so-called X_N-width Θ_N defined by

$$\Theta_N = \max_{x \in \mathscr{F}} \left(\min_{h \in \Lambda_N^{-1}} |x - x_h| \right). \tag{3.1}$$

THEOREM 5. *Suppose that $\tau \in [0, 1]$, $[\alpha] > (q + 2\tau)/2$, and $F \in \mathscr{H}$. Let $X_N = \{x_h \in \mathscr{F} \mid h \in \Lambda_N^{-1}\}$ be a $\mathscr{P}$-admissible system. Denote by $S_N^F \in \mathscr{H}$ the uniquely determined solution of the interpolation problem*

$$\int_{\mathscr{F}} |\partial^{\alpha/2} S_N^F(x)|^2 \, dV = \inf_{U \in J_N^F} \int_{\mathscr{F}} |\partial^{\alpha/2} U(x)|^2 \, dV,$$

where

$$\mathscr{J}_N^F = \{U \in \mathscr{H} \mid U(x_h) = F(x_h), h \in \Lambda_N^{-1}\}.$$

Then

$$\sup_{x \in \mathbb{R}^q} |F(x) - S_N^F(x)| \leqslant A_{\tau,\alpha} \Theta_N^\tau \left(\int_{\mathscr{F}} |\partial^{\alpha/2} F(x)|^2 \, dV \right)^{1/2},$$

where $A_{\tau,\alpha}$ is given by

$$A_{\tau,a} = \frac{4\pi}{\sqrt{\|\mathscr{F}\|}} \left[\sum_{h \notin \Lambda_M^{-1}} |\lambda_h^\alpha|^{-1} \left(|h|^{2\tau} + 2\,|h|^\tau C_\tau + C_\tau^2\right) \right]^{1/2},$$

$$C_\tau = \frac{1}{\sqrt{\|\mathscr{F}\|}} \sum_{h \in \Lambda_M^{-1}} \sum_{h' \in \Lambda_M^{-1}} |C_{h'}^h| \, |h'|^\tau,$$

and $C_{h'}^h$, $h' \in \Lambda_M^{-1}$, are the coefficients constituting B_h, $h \in \Lambda_M^{-1}$, as defined by (1.3).

Proof. For any given $x \in \mathscr{F}$, there exists a point $x_k \in \mathscr{F}$, $k \in \Lambda_N^{-1}$, with $|x - x_k| \leqslant \Theta_N$. On account of $S_N(x_k) = F(x_k)$ it is easy to see that

$$S_N^F(x) - F(x) = \mathring{S}_N^F(x) - \mathring{S}_N^F(x_k) + \mathring{F}(x_k) - \mathring{F}(x). \tag{3.2}$$

Thus, by the triangle inequality, we have

$$|S_N^F(x) - F(x)| \leqslant |\mathring{S}_N^F(x) - \mathring{S}_N^F(x_k)| + |\mathring{F}(x_k) - \mathring{F}(x)|. \tag{3.3}$$

By virtue of Theorem 2 we obtain

$$\mathring{S}_N^F(x) - \mathring{S}_N^F(x_k) = \int_{\mathscr{F}} \partial^{\alpha/2} \mathring{S}_N^F(y) \, \overline{\partial^{\alpha/2} [\mathring{K}^\alpha(y, x) - \mathring{K}^\alpha(y, x_k)]} \, dV(y). \tag{3.4}$$

Applying the Schwarz inequality we find

$$|\mathring{S}_N^F(x)-\mathring{S}_N^F(x_k)| \leqslant |\kappa^\alpha(x, x_k)|^{1/2}\left(\int_{\mathscr{F}} |\partial^{\alpha/2}S_N^F(y)|^2\,\mathrm{dV}\right)^{1/2}, \tag{3.5}$$

where we have introduced the abbreviation

$$\kappa^\alpha(x, x_k)=\int_{\mathscr{F}} |\partial^{\alpha/2}[\mathring{K}^\alpha(y, x)-\mathring{K}^\alpha(y, x_k)]|^2\,\mathrm{dV}(y). \tag{3.6}$$

S_N^F is the smoothest $\mathscr{H}$-interpolant, i.e.,

$$\int_{\mathscr{F}} |\partial^{\alpha/2}\, S_N^F(y)|^2\,\mathrm{dV} \leqslant \int_{\mathscr{F}} |\partial^{\alpha/2}F(y)|^2\,\mathrm{dV}. \tag{3.7}$$

Hence, in view of (3.3), we are able to deduce that

$$|S_N^F(x)-F(x)| \leqslant 2\,|\kappa^\alpha(x, x_k)|^{1/2}\left(\int_{\mathscr{F}} |\partial^{\alpha/2}F(y)|^2\,\mathrm{dV}\right)^{1/2}. \tag{3.8}$$

Elementary calculations yield

$$\begin{aligned}\kappa^\alpha(x, x_k) &= G^\alpha(x, x)-G^\alpha(x, x_k)-G^\alpha(x_k, x)+G^\alpha(x_k, x_k)\\ &\quad-\sum_{h\in\Lambda_M^{-1}} [\overline{B_h(x)}-\overline{B_h(x_k)}][G^\alpha(x, x_h)-G^\alpha(x_k, x_h)]\\ &\quad-\sum_{h\in\Lambda_M^{-1}} [G^\alpha(x_h, x)-G^\alpha(x_h, x_k)][B_h(x)-B_h(x_k)]\\ &\quad+\sum_{h'\in\Lambda_M^{-1}}\sum_{h\in\Lambda_M^{-1}} [\overline{B_h(x)}-\overline{B_h(x_k)}]\,G^\alpha(x_{h'}, x_h)[B_{h'}(x)-B_{h'}(x_k)].\end{aligned} \tag{3.9}$$

By use of standard arguments we are able to show that the relations

$$|\phi_h(x)-\phi_h(y)| \leqslant \frac{2}{\sqrt{\|\mathscr{F}\|}} \tag{3.10}$$

resp.

$$|\phi_h(x)-\phi_h(y)| \leqslant \frac{2\pi}{\sqrt{\|\mathscr{F}\|}}\,|h|\,|x-y| \tag{3.11}$$

hold for all $h \in \Lambda^{-1}$ and for any two $x, y \in \mathbb{R}^q$. Hence, it follows that

$$|\phi_h(x) - \phi_h(y)| \leqslant \frac{2\pi}{\sqrt{\|\mathscr{F}\|}} (|h|\,|x-y|)^\tau \tag{3.12}$$

holds for all $\tau \in [0, 1]$, $h \in \Lambda^{-1}$, and any two $x, y \in \mathbb{R}^q$. According to our assumptions we see that $|h|^\tau |\lambda_h^\alpha|^{-1/2} \in l^2(\Lambda^{-1})$. Hence, by use of (3.10), (3.12) we get from (3.9)

$$|\kappa^\alpha(x, x_k)| \leqslant \frac{4\pi^2}{\|\mathscr{F}\|} \sum_{h \notin \Lambda_M^{-1}} |\lambda_h^\alpha|^{-1} (|h|^{2\tau} + 2\,|h|^\tau C_\tau + C_\tau^2) |x - x_k|^{2\tau}. \tag{3.13}$$

Note that $A_{\tau,\alpha}$ as defined above is independent on N and $F \in \mathscr{H}$. Consequently, it follows that

$$|S_N^F(x) - F(x)| \leqslant A_{\tau,\alpha} \Theta_N^\tau \left(\int_{\mathscr{F}} |\partial^{\alpha/2} F(y)|^2 \, dV \right)^{1/2} \tag{3.14}$$

holds uniformly with respect to all $x \in \mathscr{F}$. In view of the Λ-periodicity of F, S_N^F, this implies the proof of Theorem 5. ∎

4. Uniform Approximation in C_Λ

As is well known (cf., e.g., [17]), the set of all finite linear combinations of the functins ϕ_h, $h \in \Lambda^{-1}$, is dense in the space C_Λ. Therefore, $\mathscr{H}$ is a dense subset of C_Λ too. An extended version of Helly's theorem due to Yamabe [18] states that, for any $f \in C_\Lambda$ and any ($\mathscr{P}$-admissible) system $X_N = \{x_h \in \mathscr{F} \mid h \in \Lambda_N^{-1}\}$, there exists an element $F \in \mathscr{H}$ in an ε-neighbourhood of f such that $f(x_h) = F(x_h)$ for all $h \in \Lambda_N^{-1}$. On the other hand, according to Theorem 4, any $F \in \mathscr{H}$ can be approximated uniformly to any given accuracy using spline interpolation assuming the widths Θ_N tend to zero as $N \to \infty$. Combining these results we finally arrive at the following

THEOREM 6. *Suppose that X_{N_0} is a prescribed $\mathscr{P}$-admissible system. Furthermore, let there be given a sequence (X_N) of $\mathscr{P}$-admissible systems X_N such that $X_{N_0} \subset X_N$ for every N and $\Theta_N \to 0$ as $N \to \infty$. Then, to any $f \in C_\Lambda$ and every $\varepsilon > 0$, there exist an integer $N = N(\varepsilon)$ and a spline $S \in \mathscr{S}_{\mathscr{P}}^\Lambda(X_N)$ such that $f(x_h) = S(x_h)$, $h \in \Lambda_{N_0}^{-1}$, and* $\sup_{x \in \mathbb{R}^q} |f(x) - S(x)| \leqslant \varepsilon$.

5. A Three-Dimensional Example

As an illustration our spline interpolation method will be described in more detail for the case that Λ is the three-dimensional unit lattice $\mathbb{Z}^3$ (consisting of all points in $\mathbb{R}^3$ having integral coordinates). Then, of ourse, the inverse lattice of $\mathbb{Z}^3$ coincides with $\mathbb{Z}^3$ itself. In addition, we simply choose $\rho = 0$ so that $\mathbb{Z}^3_M$ only consists of the origin.

For given data points $(x_h, F(x_h))$, $h \in \mathbb{Z}^3_N$, we discuss the interpolation problem

$$\int_{\mathscr{F}} |(-\Delta)^{\alpha/2} S^F_N(x)|^2 \, \mathrm{dV} = \inf_{U \in \mathscr{I}^F_N} \int_{\mathscr{F}} |(-\Delta)^{\alpha/2} U(x)|^2 \, \mathrm{dV}, \tag{5.1}$$

it being understood as above that $[\alpha] = \alpha_1 > 3/2$. Then the uniquely determined solution $S^F_N \in \mathscr{H}$ is given by the expression

$$S^F_N(x) = F(x_0) + \sum_{h \in \mathbb{Z}^3_N - \{0\}} a_h [G^\alpha(x, x_h) - 2G^\alpha(x, x_0) + G^\alpha(x_0, x_0)],$$

where the coefficients $a_h \in \mathbb{C}$ have to satisfy the linear equations

$$\sum_{h \in \mathbb{Z}^3_N - \{0\}} a_h [G^\alpha(x_{h'}, x_h) - 2G^\alpha(x_{h'}, x_0) + G^\alpha(x_0, x_0)]$$
$$= F(x_{h'}) - F(x_0), \qquad h' \in \mathbb{Z}^3_N - \{0\}.$$

The lattice sum

$$G^\alpha(x, y) = \sum_{\substack{|h| \neq 0 \\ h \in \mathbb{Z}^3}} \left(\frac{1}{4\pi^2 h^2} \right)^{\alpha_1} e(hx) \overline{e(hy)} \tag{5.2}$$

can be computed rapidly by a procedure due to B. R. A. Nijboer and F. W. de Wette [14]. The principle of this method can be easily seen by rewriting (5.2) formally as follows:

$$G^\alpha(x, y) = \frac{1}{\Gamma(\alpha_1)} \sum_{\substack{|h| \neq 0 \\ h \in \mathbb{Z}^3}} \frac{e(hx) \overline{e(hy)}}{(4\pi^2 h^2)^{\alpha_1}} \Gamma(\alpha_1, \pi h^2)$$
$$+ \frac{1}{\Gamma(\alpha_1)} \sum_{\substack{|h| \neq 0 \\ h \in \mathbb{Z}^3}} \frac{e(hx) \overline{e(hy)}}{(4\pi^2 h^2)^{\alpha_1}} \gamma(\alpha_1, \pi h^2) \tag{5.3}$$

(γ, Γ: incomplete gammafunctions). The first series on the right hand side has a rapid convergence whereas the second series has the same rate of convergence as the original sum. By application of the Fourier transform, however, we are able to convert this series into a rapidly convergent sum

$$G^{\alpha}(x, y)=\frac{1}{(4\pi^2)^{\alpha_1}\,\Gamma(\alpha_1)}\Bigg[\sum_{\substack{|h|\neq 0\\ h\in\mathbb{Z}^3}}\frac{e(hx)\,\overline{e(hy)}}{|h|^{2\alpha_1}}\,\Gamma(\alpha_1, \pi h^2)-\frac{\pi^{\alpha_1}}{\alpha_1}$$

$$+\pi^{2\alpha_1-3/2}\sum_{h\in\mathbb{Z}^3}\frac{\Gamma(3/2-\alpha_1, \pi\,|h-(x-y)|^2)}{|h-(x-y)|^{2\alpha_1-3}}\Bigg].$$

The expression in this form holds for $x\neq y$. For the case $x=y$, the second summation must be taken over all $h\in\mathbb{Z}^3$ with $|h|\neq 0$ and the term $\pi^{\alpha_1}/(\alpha_1-3/2)$ must be added.

The a priori estimate (Theorem 5) applied to our special situation reads

$$\sup_{x\in\mathbb{R}^3}|S_N^F(x)-F(x)|$$

$$\leqslant 2(2\pi)^{1-\alpha_1}\,|\zeta_{\mathbb{Z}^3}(2\alpha_1-2\tau)|^{1/2}\,\Theta_N^{\tau}\left(\int_{\mathscr{F}}(-\Delta)^{\alpha/2}\,F(x)|^2\,dV\right)^{1/2}$$

assuming that $\tau\in(0, 1]$ and $\alpha_1>(3+2\tau)/2$, where $\zeta_{\mathbb{Z}^3}$ is the zeta function given by

$$\zeta_{\mathbb{Z}^3}(s)=\sum_{\substack{|h|\neq 0\\ h\in\mathbb{Z}^3}}\frac{1}{|h|^s}. \tag{5.4}$$

In particular, for $\tau=1$ and $F=\phi_h$, $|h|\neq 0$, we obtain

$$\sup_{x\in\mathbb{R}^3}|S_N^{\phi_h}(x)-\phi_h(x)|\leqslant 4\pi\,|\zeta_{\mathbb{Z}^3}(2\alpha_1-2)|^{1/2}\,\Theta_N\,|h|^{\alpha_1}. \tag{5.5}$$

In order to get a quantitative impression of the accuracy we finally evaluate the zeta sum for some different orders:

α_1	$\zeta_{\mathbb{Z}^3}(2\alpha_1-2)$
3	1.653232 E + 1
4	8.401924 E + 0
5	6.945808 E + 0
6	6.426120 E + 0

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